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IN WHAT PROOF WOULD A GEOMETER USE THE *ΠΟΔΙΑΙΑ*?*

In memoriam Filippo Càssola

A *ποδιαία* is a straight line one foot long. Aristotle refers to it in five passages scattered among the *Analytica* and the *Metaphysica*.[1] The point Aristotle makes in three of these texts is that incidental or material features of the mathematical objects set out in a diagram by the geometers do not lead them to assert something false. The first passage is at *An. pr*. 49b33–7 (text **1**):

Οὐ δεῖ δ' οἴεσθαι παρὰ τὸ ἐκτίθεσθαί τι συμβαίνειν ἄτοπον· οὐδὲν γὰρ προσχρώμεθα τῷ τόδε τι εἶναι, ἀλλ' ὥσπερ ὁ γεωμέτρης τὴν ποδιαίαν καὶ εὐθεῖαν τήνδε καὶ ἀπλατῆ εἶναι λέγει οὐκ οὔσας, ἀλλ' οὐχ οὕτως χρῆται ὡς ἐκ τούτων συλλογιζόμενος.

At *An. post*. 76b39–77a3 (**2**), Aristotle is explaining the difference between a term and a hypothesis, and introduces a rather incidental remark about the role of hypotheses in geometry.

οὐδ' ὁ γεωμέτρης ψευδῆ ὑποτίθεται, ὥσπερ τινὲς ἔφασαν, λέγοντες ὡς οὐ δεῖ τῷ ψεύδει χρῆσθαι, τὸν δὲ γεωμέτρην ψεύδεσθαι λέγοντα ποδιαίαν τὴν οὐ ποδιαίαν ἢ εὐθεῖαν τὴν γεγραμμένην οὐκ εὐθεῖαν οὖσαν. ὁ δὲ γεωμέτρης οὐδὲν συμπεραίνεται τῷ τήνδε εἶναι γραμμὴν ἣν αὐτὸς ἔφθεγκται, ἀλλὰ τὰ διὰ τούτων δηλούμενα.

In this passage, then, the geometer not only describes some geometrical objects, but also makes some assumptions (*ὑποτίθεται*) involving them. The three characteristics of the line (namely, being one foot long, straight and breadthless) proposed in **1** are reduced to two. Passage **1** raises some interpretative problems, mainly related to the meaning of *ἐκτίθεσθαι*. I shall not address that question, which has been sufficiently debated in the literature. I shall focus on the Aristotelian claims that a geometer calls a certain line one foot long even if the line is not so, and that positing a *ποδιαία* is on a par with other geometrical assumptions. The latter statement has bewildered some commentators: they rightly ask themselves in what proof a geometer would use such a premiss,[2] which appears, as it does, to carry a material connotation that is *prima facie* unsuited to a geometrical argument. I shall answer the question, showing, on the one hand, that a very peculiar mathematical practice fits the Aristotelian claim very well, and on the other hand, that the 'geometer' referred to can be given a name.

* I thank an anonymous referee for remarks.

[1] I shall refer to the following commentaries by the name of the modern commentator and the date of publication: *Aristotle's Metaphysics*. A Revised Text with Introduction and Commentary by W.D. Ross (Oxford, 1924); *Aristotle's Prior and Posterior Analytics*. A Revised Text with Introduction and Commentary by W.D. Ross (Oxford, 1949); T.L. Heath, *Mathematics in Aristotle* (Oxford, 1949); Aristotele, *Gli Analitici Primi*. A cura di M. Mignucci (Napoli, 1969); Aristotle, *Prior Analytics*. Translated, with introduction, notes, and commentary, by R. Smith (Indianapolis/Cambridge, 1989); Aristotle, *Posterior Analytics*. Translated with a Commentary by J. Barnes (Oxford, 1993[2]).

[2] So Barnes 1993, 143, when commenting on passage **2**. See also H. Mendell, 'Making sense of Aristotelian demonstration', *OSAP* 16 (1998), 161–225, at 182.

The other passages in which the simile of the geometer positing a *ποδιαία* occurs are analogous to the ones just cited; the discrepancies are not essential to our purposes but nevertheless worth mentioning.

At *Metaph.* 1089a21–5 (**3**) Aristotle is writing against the Platonic tenet that falsehood (a form of not-being) is necessary for multiplicity to come to be. The simile is the same as in **2**, but only the *ποδιαία* survives in it:

διὸ καὶ ἐλέγετο ὅτι δεῖ ψεῦδός τι ὑποθέσθαι, ὥσπερ καὶ οἱ γεωμέτραι τὸ ποδιαίαν εἶναι τὴν μὴ ποδιαίαν· ἀδύνατον δὲ ταῦθ' οὕτως ἔχειν, οὔτε γὰρ οἱ γεωμέτραι ψεῦδος οὐθὲν ὑποτίθενται (οὐ γὰρ ἐν τῷ συλλογισμῷ ἡ πρότασις).

The polemical target in passages **2** and **3** is usually identified with Protagoras.[3]

At *Metaph.* 1078a17–21 (**4**) the context is the opposite of the one in the preceding passages: subordinate sciences, such as harmonics, optics, and mechanics, make statements about the real world, but their investigations are framed in geometrical language. Therefore, the incidental features of the objects of study are not taken into account. In spite of this, those sciences are able to make true statements about their real objects of inquiry. In Aristotle's words (the objects are simply posited, as in **1**):

ὥστ' εἴ τις θέμενος κεχωρισμένα τῶν συμβεβηκότων σκοπεῖ τι περὶ τούτων ᾗ τοιαῦτα, οὐθὲν διὰ τοῦτο ψεῦδος ψεύσεται, ὥσπερ οὐδ' ὅταν ἐν τῇ γῇ γράφῃ καὶ ποδιαίαν φῇ τὴν μὴ ποδιαίαν· οὐ γὰρ ἐν ταῖς προτάσεσι τὸ ψεῦδος.

In order to make the simile of the *ποδιαία* fit the context, one must suppose that the geometer is interested in investigating the properties of the concrete line he has drawn, but no falsehood arises if he sets up a geometrical proof by assuming that the line has a certain well-defined length, and therefore is an abstract object (since no material object can be exactly one foot long). This interpretation is rather contrived and the intended meaning is at variance with the one at work in passages **1** to **3**. More likely, then, the simile is here ill-chosen, and Aristotle is simply resorting to a stock example of the abstract/concrete polarity in geometry.

A short allusion at *Metaph.* 1052b31–3 (**5**) can be set in parallel with the above passages. Aristotle has just asserted that a standard measure of length, breadth, depth, weight, or speed must be taken as an undivided unit. As an afterthought, he brings in geometry:

ἐν πᾶσι δὴ τούτοις μέτρον καὶ ἀρχὴ ἕν τι καὶ ἀδιαίρετον, ἐπεὶ καὶ ἐν ταῖς γραμμαῖς χρῶνται ὡς ἀτόμῳ τῇ ποδιαίᾳ.

The reference to geometry is puzzling, because asserting that geometers make use of a line as undivided is near to anathema, yet it has not received much attention by the commentators.[4]

In all of the above passages, it seems plain that Aristotle is alluding to the fact that a drawing cannot exactly represent a geometrical object. A first problem with the rather concise Aristotelian formulation is whether the geometer positing the *ποδιαία* wants his drawing to be as accurate as possible (obviously falling short of absolute precision), or draws a line at random and for some obscure reason calls it a *ποδιαία*. In the first case, the simile would be portraying a geometer performing proofs in the

[3] On the basis of e.g. *Metaph.* 997b35–998a4.

[4] In Ross 1924 the *ποδιαία*-clause is omitted from the paraphrasis, and no commentary is provided.

metrical or land-surveying domain, as for instance those transmitted in the Heronian *corpus*. This would easily account for the activity of positing a ποδιαία and investigating its properties or the consequences of positing it, as if being a ποδιαία were a crucial feature of the line. The second possibility entails instead that being a ποδιαία is an incidental, and perhaps a conventional, feature of the posited line. In this case the Aristotelian reference is more opaque. The texts suggest that the meaning intended by Aristotle is the second one, as the similes refer to other abstract features of a line. Passage **4** seems to contradict this reading, as it adds a realistic element: 'one draws on the ground and calls a foot long the <line> that is not a foot long'. But it is well known that ancient geometry was mainly performed by drawing the diagrams in the sand, on a horizontal plane surface lying on the ground. Therefore, Aristotle is referring by metaphor to the geometer's activity of producing proofs, and not to some mensuration performed for land-surveying purposes. Tracing a ποδιαία on the ground is, paradoxically enough, the sign that the line was introduced in actual and fully rigorous proofs,[5] possibly involving the highest degree of generality.

A further point to be made is that the geometers' activity involved in the simile can by no means be made to correspond to the instantiation (ἔκθεσις) of the geometrical objects involved in a proof,[6] an identification purportedly fitting Alexander's interpretation of the ἐκτίθεσθαι-procedure referred to in passage **1**.[7] In this reading, positing a ποδιαία becomes something like a debased version of asserting ἔστω εὐθεῖα γραμμὴ ἡ *ΓΔ*, a typical assumption occurring in a geometrical instantiation. That this identification cannot stand is made clear by the other features of the line posited by the geometer in passages **1** and **2**, namely, being straight and breadthless.[8] As a consequence, Aristotle says that the *abstract* feature of being *exactly* one foot long is what is upset by any proof made by a working mathematician, since no line drawn in the sand can be exactly one foot long: the ποδιαία is on the same side, namely, the abstract one, as the straight line and the breadthless line. What happens in a geometrical instantiation is far removed from what Aristotle asserts concerning the ποδιαία: no material features are there at issue,[9] and one should disentangle once and for all the procedure of instantiation from the representation of the proof by means of a geometrical diagram (and not only from the material features of the latter).[10]

After these preliminaries, we come to our question: what mathematical practice might possibly have introduced lines one foot long as an integral part of a geometrical proof? After all, geometry is hopefully dealing with general objects, or at least with generic individuals, of which a ποδιαία can hardly be a representative specimen. A

[5] In e.g. text **5** this is borne out by the specification ἐν ταῖς γραμμαῖς, a phrase that canonically denotes rigorous geometrical proofs involving diagrams.

[6] *Pace* Mignucci 1969, 495, Ross 1924, 476, Ross 1949, 541, Mendell (n. 2), 181–2, and, from a slightly different point of view, Smith 1989, 173.

[7] Alexander asserts that the procedure alluded to by Aristotle is the use of denotative letters in schematic expositions of syllogistic (*In An. pr.* 379.14–380.27).

[8] Actually, no geometer would explicitly posit the latter feature of a line in a theorem. Nevertheless, being breadthless is an essential attribute of a line, since the latter is defined as a 'breadthless length' (cf. *El.* 1.def.2; the definition was known by Aristotle, who discusses it in an anti-Platonic perspective at *Top.* 143b11–32). As a consequence, Aristotle rightly claims that a geometer is implicitly positing such a feature in each proof involving lines.

[9] The curious belief that in the instantiation something might be wrong because of the mere act of instantiating is apparently endorsed in Heath 1949, 219.

[10] In fact, the problems Aristotle is pointing out by means of his simile arise when the geometer refers to material features of a concrete diagram, not to the abstract array of lines representing a geometrical state of affairs.

clue to the answer comes if one looks at the use in the *Elements* of the verb ἐκτίθημι, that obviously plays a key role in passage **1**. The verb itself is poorly attested in the Euclidean *corpus*,[11] but there are recorded 174 occurrences of forms of ἔκκειμαι that was currently used as the passive of ἐκτίθημι. Most of the occurrences (121 items) are included in propositions in Book 10 of the *Elements*. In this book, containing 115 propositions as well as several lemmas and porisms, a classification of irrational lines is provided. An irrational line is a straight line that is incommensurable (in a rather complex sense to which we shall return later) with a straight line assigned as a reference in advance. The reference line is called 'expressible' (ῥητή) in the *Elements*,[12] and is an essential ingredient in any proof about irrational lines. The assumption that a suitable ῥητή is posited is repeated in the majority of theorems in Book 10, simply because the expressible line suitable to the purposes of each single proof varies from theorem to theorem. A typical way of formulating this assumption is ἐκκείσθω ῥητὴ ἡ *AB*. Forms of the imperative of ἔκκειμαι are used, and not of εἰμί as usual in an instantiation, since the expressible line is seldom among the objects mentioned in the general statement enunciating the theorem: for this reason, the ἐκκείσθω-assumption is usually made in the body of the proof, and not in the instantiation proper.[13] In a sense, then, the choice of the expressible line is left to the mathematician responsible for the proof, a move that operates on a different level from those usually performed in an instantiation. In the few occurrences in which the expressible is mentioned in the enunciation as a given, the imperative ἐκκείσθω instead of ἔστω is adopted in the instantiation itself, a remarkable exception to the standard practice in the *Elements*.

As an exemplification of this description we may read the beginning of *El.* 10.60:

Τὸ ἀπὸ τῆς ἐκ δύο ὀνομάτων παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πρώτην.
Ἔστω ἐκ δύο ὀνομάτων ἡ ΑΒ διῃρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ, ὥστε τὸ μεῖζον ὄνομα εἶναι τὸ ΑΓ, καὶ ἐκκείσθω ῥητὴ ἡ ΔΕ, [...]

In other contexts, the objects marked by ἐκκείσθω are usually subject to stricter constraints than other entities selected as given. In the domain of number theory, for instance, assignments are met such as ἀπὸ μονάδος ἐκκείσθωσαν ὁσοιδηποτοῦν ἀριθμοὶ ἐν τῇ διπλασίονι ἀναλογίᾳ (*El.* 9.36).[14] This means that the sequence of integers 1, 2, 4, 8, 16, ... must be set out until some condition is met. In the late arithmetical tradition, represented for instance by the works of Nicomachus and Iamblichus,[15] this kind of assumption became the standard formulation by which concrete sequences of numbers were introduced. Therefore, the ἐκκείσθω-assumption often carries a strongly particularising connotation.

[11] Only 3 occurrences, in *El.* 9.36, 13.18 and in the alternative proof of 10.23. The noun ἔκθεσις scores 2 occurrences, contained in two alternative proofs of redaction **b** of the *Phaenomena*.

[12] The adjective ῥητός is already attested in the Platonic *corpus* as a technical term (e.g. in the mathematical passage at *Resp.* 546C), but it does not have the precise meaning it has in the *Elements*, nor can it be taken to refer to any mathematical object.

[13] As a consequence, the ἐκκείσθω-assumption is not to be confused with the instantiation itself, i.e. what came later to be (quite improperly) called ἔκθεσις. I shall discuss elsewhere the differences between the ἔκθεσις of the mathematicians and the ἔκθεσις of the philosophers.

[14] In the enunciation of this theorem we find one of the three occurrences of forms of ἐκτίθημι, as seen above.

[15] The three standard forms ἔκθεσις, ἐκτίθημι, ἔκκειμαι occur almost equally in Nicomachus' *Introductio arithmetica* and Iamblichus' commentary on it, with slight prominence accorded to the substantive.

No theorems in Book 10 of the *Elements* posit a ποδιαία. Nevertheless, the theory of irrational lines there developed is the only piece of abstract Greek geometry where a reference line, fairly independent from the givens of the proof at issue, is set out as a standard against which to check the 'size' of other lines. Since it is unlikely that the theory as we read it in *El.* 10 was already in its final form in Aristotle's time, the absence of the ποδιαία cannot be taken to entail that that term was not used earlier for this line in just this context. That some theory of irrational lines was already at hand when Aristotle wrote the above passages is borne out by a series of testimonies that I shall not discuss here. However, the most important of these testimonies nicely fits the point I am going to argue. The reader should have already suspected that Aristotle's reference is in essence a literary one, and that the geometer at work with the ποδιαία is Theodorus, engaged in a series of proofs about incommensurable lines at *Tht.* 147D:[16]

Περὶ δυνάμεών τι ἡμῖν Θεόδωρος ὅδε ἔγραφε, τῆς τε τρίποδος πέρι καὶ πεντέποδος ἀποφαίνων ὅτι μήκει οὐ σύμμετροι τῇ ποδιαίᾳ, καὶ οὕτω κατὰ μίαν ἑκάστην προαιρούμενος μέχρι τῆς ἑπτακαιδεκάποδος· ἐν δὲ ταύτῃ πως ἐνέσχετο.

We find here the only occurrence of the noun ποδιαία in the Platonic *corpus*, and this welcome fact, in my opinion, makes the identification certain. There is no reason to suppose that Plato's short description is unfaithful to the actual status of the theory of irrational lines around 399 B.C. (the dramatic date of the dialogue) or in the few decades following that date. What Theodorus does is to fix a reference line, namely, a ποδιαία, and to use it as a standard against which the other lines have to be set as far as their commensurability is concerned. Of course, the actual line Theodorus drew in the sand was not exactly a one-foot-long straight line, but nevertheless the truth of the conclusions he draws is unaffected by this fictitiously particularising feature – and this is exactly what Aristotle asserts. On the other hand, fixing a reference line such as the ποδιαία is a necessary mathematical step if one wants to establish whether other lines are (in)commensurable or not. In fact, (in)commensurability is a relational concept: for instance, using Theodorus' terminology, the twelve-foot-power is commensurable with the three-foot-power (actually, it is simply the double of it), and if we take the latter as a reference, the former could not be a δύναμις in the sense of Theaetetus' definition given as a consequence of Theodorus' lesson. Moreover, Theaetetus' distinction itself at *Tht.* 148A6–B2 between μήκη and δυνάμεις entails that the reference line was in fact taken, by him and by Theodorus before him, as a minimal reference measure,[17] and that being commensurable/incommensurable with it is the only allowed polarity.[18] The point, then, is that to classify irrational lines some reference line has to be fixed, and if, on the one hand, it is immaterial whether we call

[16] There has been an extensive debate among scholars as to whether the δυνάμεις in the Platonic passage are lines or plane domains. They are lines, and expressions such as e.g. τῆς τρίποδος are very abbreviated phrases standing for 'the line capable of a domain of three square feet'. Such a domain is most conveniently represented as a rectangle whose sides are a three-foot-long straight line and a ποδιαία; in a sense, then, the rectangle is completely determined by the three-foot-long side only, and the elliptical denomination is fully justified.

[17] The reference line enters only indirectly in Theaetetus' classification, which places all lines that 'square equilateral and plane *numbers*' in the category of μῆκος. These 'numbers' are in fact the result of actually measuring straight lines assuming a ποδιαία as unit.

[18] Notice that the specification μήκει (*Tht.* 147D4 and 148B1) qualifies only the relation 'being *not* commensurable', and simply marks the distinction between (in)commensurability of lines and (in)commensurability of plane domains.

it, and posit it as, a ποδιαία or a δακτυλιαία, on the other hand, the line must have a fixed and well-defined length as its essential feature, since it is exactly on this feature that the determination of which among the other lines is a δύναμις depends. As a consequence, for the purposes of the theory of irrational lines the reference line may well be taken as 'undivided' (cf. passage **5**). However, that the reference line is an actual unit of measure (a conception of the reference line we may call 'metrological') is not a necessary feature,[19] and this might well have induced doubts about the generality of the whole procedure. At least, from the Aristotelian passages we may gather that doubts were raised, if such a generality had to be expressly pointed out.

The 'metrological' conception of the reference line is at variance with the one at work in *El.* 10 and 13, where no measuring takes place. In the *Elements*, in fact, two kinds of commensurability for straight lines are introduced (10.def.1–3): straight lines are commensurable in length (μήκει) when they admit a common measure; straight lines are commensurable in power (δυνάμει) when the squares on them admit a common measure.[20] It is easy to construct lines that are commensurable in power only, but, once a reference straight line has been selected as the ῥητή, not only lines commensurable in length (and hence in power too) with the ῥητή but also lines commensurable in power only with it are called ῥηταί (*El.* 10.def.3). Irrational lines are those that are incommensurable in power (and hence in length too) with the ῥητή. A ῥητή is a provisional reference line, and must be fixed anew in every theorem. In a 'metrological' conception, commensurability *tout court* is instead identified with commensurability in length, and the reference line is fixed once and for all.[21] As a consequence, none of the lines envisaged by Theodorus could be termed 'irrational' in the sense of the *Elements* (even if Theodorus only asserts that they are not commensurable in length with the ποδιαία), and, were the ποδιαία taken to be the ῥητή, they would be all ῥηταί as well.[22] If the interpretation of the Aristotelian allusions proposed in this paper is right, those allusions and the *Theaetetus* passage suggest that the metrological notion was the one at work in the pre-Euclidean theory of irrational lines. The reference line was probably fixed by taking it to be a ποδιαία, a move that must be read as entirely conventional and hence puts that line in the same domain of arbitrary individuals where the lettered geometrical objects set out in a proof belong. This is the reason why Aristotle ranges his ποδιαία with such abstract entities as a straight line and a breadthless line. Besides changing the very conception of reference line, the author of the theory that was eventually embodied in the *Elements* got rid of

[19] Cf. also *Leg.* 820C, where commensurability and incommensurability are formulated in terms of objects μετρητά (resp. ἄμετρα) πρὸς ἄλληλα.

[20] As a consequence, the expressible straight line can even be larger than the irrational lines constructed with reference to it, which is clearly impossible according to the 'metrological' approach. A case in point is provided by propositions 13.13–17, where the expressible set out is the diameter of a sphere in which the regular polyhedra have to be inscribed, and the related irrational lines are the edges of the inscribed polyhedra.

[21] The peculiar Euclidean conception of expressibility caused misgivings in the majority of commentators: in Hero, Pappus, in several scholia to Book 10, and in the Arabic commentators attempts are made at reducing the Euclidean notion to the 'metrological' one. Among the examples of reference lines provided in some commentaries, the ποδιαία is often mentioned. A list of this kind found its way in the very text of *El.* 10.def.3 carried by the Theonine manuscripts.

[22] In the terminology of the *Elements*, the δυνάμεις introduced by Theaetetus would simply be lines commensurable in power only with the ποδιαία. Notice also a crucial grammatical shift: in Theaetetus' classification, the terms μῆκος and δύναμις are substantives denoting mathematical objects; in the *Elements* only the datives of respect μήκει and δυνάμει appear, as determinatives of a relation.

the residual material connotation lying in the use of the term ποδιαία, and replaced it with ῥητή. However, the value of the Aristotelian passages as *independent* testimony about the pre-Euclidean theory of irrational lines should not be overestimated. If we accept the dependence on the *Theaetetus* passage as likely, then we cannot conclude that fixing a ποδιαία was still a current practice when Aristotle made his allusions to it. On the other hand, the repeated use of the simile of the ποδιαία, and the association of it with other, basic features of a line not connected with irrationality, might suggest that a composite *topos* was developed, maybe as an answer to objections of a Protagorean kind about an alleged loss of generality in actual geometrical proofs. In this case, a direct connection with the *Theaetetus* passage would be lost, but there is no warranty that the *topos* is an Aristotelian elaboration. It is not clear, then, whether the elimination of the residual material connotation lying in the use of the ποδιαία occurred before or after Aristotle, or even as a consequence of his repeatedly referring to it.

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